Note that $\bar{E}_n(q, r)$ $[\bar{F}_n(q, r)]$ is not the Laplace transform of $E_n(\beta, r)[F_n(\beta, n)]$.

In these formulas (41–43), β is used to refer to a particular eigenvalue $q = \beta_{n,j}$, satisfying (39); the temperature (41) involves a sum over all these eigenvalues ($j = 1, 2 \ldots,$ $n = 0, 1, \ldots$). In an earlier paper, Warren[1] gives a solution to this problem. The present method has computational advantages over Warren's solution, particularly in the neighborhood of the inner boundary $r = b$ and for small times. Indeed Warren's approach converges nonuniformly near the inner boundary $r = b$ and takes the value zero at that boundary rather than the nonhomogeneous value of Eq. (32).

References

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A Steepest-Ascent Solution of Multiple-Arc Optimization Problems

Rollyn G. Graham*
Aerospace Corporation, El Segundo, Calif.

Nomenclature

$f^{(a)}$	=	n-dimensional vector of known functions of x, u, and t
g	=	gravitational acceleration
h	=	altitude
m	=	mass
r	=	range
T	=	thrust magnitude
t	=	independent variable, time
t_a	=	corner times
u	=	r-dimensional vector of steering functions
v	=	horizontal velocity
w	=	vertical velocity
x	=	n-dimensional vector of state variable histories
β	=	mass flow rate
(Δt)	=	A-dimensional vector of corrections to the corner times
Δz	=	s-dimensional vector of time invariant control variable corrections
θ	=	thrust angle
λ	=	n-dimensional vector of Lagrange multipliers
φ	=	performance index
Ω	=	q-dimensional vector of constraint functions, $q \leq n$

The problem of trajectory optimization has received a great deal of attention in recent years. The problem, as usually stated, involves the determination of one or more steering functions, such as angle of attack or throttle setting, such that some performance index is optimized, subject to specified constraints. One of the most successful approaches to this problem has been the so-called steepest-ascent or gradient technique, developed by Bryson and Denham[1,2] and Kelley.[3] The problems that have been treated with this approach, however, are actually special cases in that they involve only one subarc. This note extends the steepest-ascent technique for the simultaneous optimization of time dependent functions and time invariant quantities, originally mentioned by Denham,[2] to multiple-arc problems with the associated unknown corner times.

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* Member of the Technical Staff, Performance Analysis Department. Associate Member AIAA.

The technique mentioned in Ref. 2 is applicable to the class of problems for which small variations in the constraint functions and in the performance index are related to small changes in the control variables (either time dependent or invariant) in the following manner:

$$\Delta\varphi = A_\varphi^T \Delta z + \int_{t_0}^{t_A} \lambda_{u\varphi}^T \, \delta u \, dt \tag{1}$$

$$\Delta\Omega = A_\Omega^T \Delta z + \int_{t_0}^{t_A} \lambda_{u\Omega}^T \, \delta u \, dt \tag{2}$$

In order to apply the technique of Ref. 2, the multiple-arc problem must be expressed in the form of Eqs. (1) and (2).

For general multiple-arc problems, the governing set of differential equations for the state variables can be written as follows:

$$J^{(a)} = \dot{x}^{(a)} - f^{(a)}(x, u, t) = 0 \qquad \text{for } t_{a-1} \leq t < t_a, \ a = 1, \ldots A \tag{3}$$

A complete solution to the differential equations consists of a number of segments. Each segment is referred to as a subarc, and the juncture of two subarcs is referred to as a corner. The length and position of a subarc are defined by the corner times; in particular, the ath subarc begins at t_{a-1} and ends at t_a. It is assumed that either a priori knowledge or some logical procedure is available to determine the number of subarcs in the solution.

The superscript (a) notation in Eq. (3) is used because the expressions governing the derivatives of the state variables x are not necessarily continuous from one subarc to the next. Discontinuities can result from a change in the algebraic form of the derivatives, from a discontinuity in u, or from a discontinuity in some parameter in the system.

For the present, it is assumed that t_0 and all initial conditions for the differential equations are given. The following constraints must be satisfied at the terminal point t_A:

$$\Omega = \Omega[x(t_A), t_A] = 0 \tag{4}$$

Subject to the differential constraints (3) and the terminal constraints (4) it is desired to optimize som performance index:

$$\text{performance index} = \varphi[x(t_A), t_A]$$

Because of the existence of corners, the standard relationship between the adjoint system and the linearized form of Eq. (3) is not directly applicable. An analogous result can be derived, however, which does consider the existence of corners and the associated discontinuities in the derivatives. By predicting the effects of small changes in the corner times on the terminal conditions, this analogous result permits a logical correction procedure that will ultimately yield an optimal multiple-arc solution.

Before proceeding with the derivation, one comment on notation must be made. The derivative expressions $f^{(a)}$ are defined for $t < t_a$ but are not rigorously defined for $t = t_a$. It is assumed, however, that

$$\lim_{t \to t_a} f^{(a)}(x, u, t) \quad \text{for } t_{a-1} \leq t < t_a$$

is well-behaved. Whenever it is indicated that $f^{(a)}$ is evaluated at t_a, it should be understood to mean the limit as t approaches t_a.

The derivation of the desired relationship begins with the formation of the following integral:

$$F = \sum_{a=1}^{A} \int_{t_{a-1}}^{t_a} \lambda^T J^{(a)} \, dt \tag{5}$$

Small variations in x, u, and t_a are now introduced, and only first-order terms are retained:

$$\Delta F = \sum_{a=1}^{A} \int_{t_{a-1}}^{t_a} \lambda^T \, \delta J^{(a)} dt + \sum_{a=1}^{A-1} [\lambda^T (J^{(a)} - J^{(a+1)})]_{t_a} \Delta t_a + [\lambda^T J^{(A)}]_{t_A} \Delta t_A = 0 \tag{6}$$

By expanding $J^{(a)}$ in a Taylor series, integrating by parts, and using the relationship between variations at fixed time points and total variations, the following result is obtained:

$$\Delta F = \sum_{a=1}^{A} (\lambda^T \Delta x)_{t_{a-1}}^{t_a} - \sum_{a=1}^{A-1} [\lambda^T(f^{(a)} - f^{(a+1)})]_{t_a} \Delta t_a - (\lambda^T f^{(A)})_{t_A} \Delta t_A - \sum_{a=1}^{A} \int_{t_{a-1}}^{t_a} \lambda^T \frac{\partial f^{(a)}}{\partial u} \delta u \, dt - \sum_{a=1}^{A} \int_{t_{a-1}}^{t_a} \left(\dot\lambda + \left[\frac{\partial f^{(a)}}{\partial x} \right]^T \lambda \right)^T \delta x \, dt \quad (7)$$

In order to eliminate the last term, the multiplier functions λ are specified as solutions to the adjoint equations:

$$\dot\lambda = -\left[\frac{\partial f^{(a)}}{\partial x}\right]^T \lambda \qquad t_{a-1} \leq t < t_a \qquad a = 1, \ldots A \quad (8)$$

The influence functions are defined as:

$$\lambda_u = \left[\frac{\partial f^{(a)}}{\partial u}\right]^T \lambda \qquad t_{a-1} \leq t < t_a \qquad a = 1, \ldots A \quad (9)$$

Substituting (8) and (9) into (7), and assuming that x is continuous† across corners provides the desired relationship:

$$(\lambda^T \Delta x)_{t_A} - (\lambda^T \delta x)_{t_0} - \sum_{a=1}^{A-1} [\lambda^T(f^{(a)} - f^{(a+1)})]_{t_a} \Delta t_a - (\lambda^T f^{(A)})_{t_A} \Delta t_A = \int_{t_0}^{t_A} \lambda_u^T \delta u \, dt \quad (10)$$

Equation (10) is an extension to the case with multiple subarcs of the standard relationship between linear differential equations and the associated adjoint set.

By the appropriate choice of boundary conditions for λ at $t = t_A$, the form of Eq. (10) can be made identical to that of Eqs. (1) and (2). For those cases without corners, the boundary conditions on λ are usually chosen such that the Δt_A term of Eq. (10) is cancelled. Since corrections to the corner times are the essence of this report, however, t_A is used as the stopping condition on each trajectory; its value is corrected after each iteration using the generalized steepest-ascent technique. The following sets of boundary conditions on λ are defined for the current problem:

$$\lambda_\varphi^T(t_A) = [\partial\varphi/\partial x]_{t_A} \quad (11)$$

$$\lambda_\Omega^T(t_A) = [\partial\Omega/\partial x]_{t_A} \quad (12)$$

Combining Eqs. (10) and (12) and assuming fixed initial conditions, one obtains equations identical to (1) and (2) where the following definitions apply:

$$A_\Omega^T = \begin{bmatrix} a_{11} & \ldots & a_{1A} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{q1} & \ldots & a_{qA} \end{bmatrix} \qquad A_\varphi^T = [a_{01}\, a_{02} \ldots a_{0A}]$$

$$a_{0a} = \{\lambda_\varphi^T(f^{(a)} - f^{(a+1)})\}_{t_a} \qquad a = 1, \ldots A\text{-}1 \quad (13)$$

$$a_{0A} = \{\lambda_\varphi^T f^{(A)} + \partial\varphi/\partial t\}_{t_A} \quad (14)$$

$$a_{ka} = \{\lambda_{\Omega_k}^T(f^{(a)} - f_a^{(a+1)})\}_{t_a} \qquad \begin{matrix} k = 1, \ldots q \\ a = 1, \ldots A\text{-}1 \end{matrix} \quad (15)$$

$$a_{kA} = \{\lambda_{\Omega_k}^T f^{(A)} + \partial\Omega_k/\partial t\}_{t_A} \qquad k = 1, \ldots q \quad (16)$$

Using the generalized steepest-ascent technique, one can now proceed in an orderly fashion to an approximate optimum multiple-arc solution.

As an application of the technique, the problem of maximizing the range of a rocket operating over an airless, flat moon is considered. The terminal constraints specify a final weight and a soft landing at a given altitude. The equations of motion are as follows:

$$\dot x_1 = \dot v = (T/m)\cos\theta = f_1 \quad (17)$$

$$\dot x_2 = \dot w = (T/m)\sin\theta - g = f_2 \quad (18)$$

$$\dot x_3 = \dot r = v = f_3 \quad (19)$$

$$\dot x_4 = \dot h = w = f_4 \quad (20)$$

$$\dot x_5 = \dot m = -\beta = f_5 \quad (21)$$

During the thrusting subarcs, T and β are nonzero constants; during the coasting subarc, they are zero. The boundary conditions that must be satisfied at the terminal time are as follows:

$$\Omega = \begin{Bmatrix} v(t_3) - v^* \\ w(t_3) - w^* \\ h(t_3) - h^* \\ m(t_3) - m^* \end{Bmatrix} = 0 \quad (22)‡$$

The performance index to be maximized is

$$\varphi = r(t_3) \quad (23)$$

The pertinent boundary conditions and constants are as follows: $v(t_0) = 0$; $w(t_0) = 0$; $r(t_0) = 0$; $h(t_0) = 0$; $m(t_0) = 10{,}000$ lbm; $T = 2{,}000$ lb; $\beta = 8$ lbm/sec; $g = 5.3$ ft/sec². The optimum steering program, as determined by the steepest-ascent procedure, is presented in Fig. 1. The procedure required approximately 20 iterations on an IBM 7090.

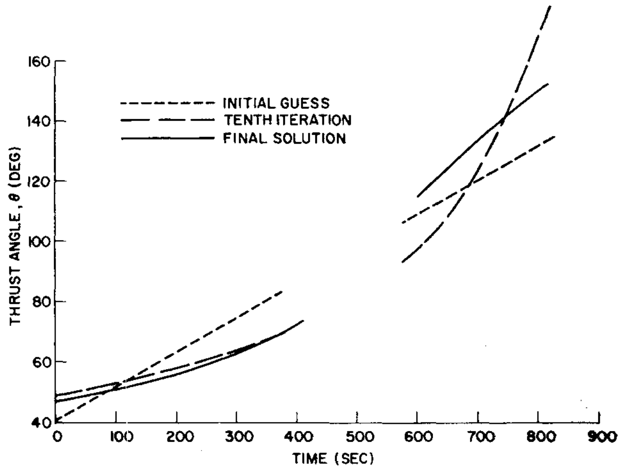

Fig. 1 Thrust angle vs time.

In conclusion, the author feels that the technique described herein opens up some interesting avenues of investigation. Most of these avenues are as yet unexplored, but a procedure that is at least theoretically workable does exist. The procedure has been shown to be workable in practice for some cases, and it is thought that similar results can be obtained for many other cases.

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† The general case involving discontinuous state variables is developed in Ref. 4.

‡ $v^* = 0$: $w^* = 0$: $h^* = 0$: $m^* = 5000$ blm.